Math 2001 Final Exam
2:00-5:00pm December , 2019

Full name ....

Signature ....

Student ID ....

- Closed book, closed notes; No calculator; No cell phone;
- Please read each problem carefully and show all your work.
- *Writings on the back side of the papers would* ***not*** *be graded but you may use these for your own scratch work.*
  Write your solutions on the front side of the pages.
- There are 105 points possible on this test.

1. (24 points) State the following definitions

   a) Using only logical symbols, variables, and mathematical symbols, state the definition of "x is the least upper bound for the set A."

   b) State the definition that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = c$.

   c) Give two equivalent formulations of the statement that f is continuous at $x = c$.

   d) State the definition that $\lim_{x \to a} g(x) = k$.

Definitions (continued)

e) State the definition of f is differentiable at $x = c$.

f) State the definition that x is the $\limsup$ of the sequence (s_n).

g) Give an example of a sequence (s_n) where $\limsup(s_n) = \liminf(s_n)$ and a sequence (t_n) where $\limsup(t_n) > \liminf(t_n)$.

h) Give the definition of the sequence (a_n) is a Cauchy sequence.

2. (10 points) Suppose that (s_n) is a sequence. Prove if $\lim s_n = \infty$ then $\lim \frac{1}{s_n} = 0$. Recall that $\lim s_n = \infty$ means that for every $M > 0$ there is N such that $s_n > M$ for all $n > N$.

3. (16 points) Below and on the next page is a proof of the **Theorem**: *If every sequence in X has a convergent subsequence whose limit is in X, then X is compact.* Answer the questions below related to this proof.

Proof. First we show that X must be bounded. By way of contradiction, suppose that X is not bounded above. Then for each natural number choose $x_n \in X$ such that $x_n > n$

(a) Explain why there is for each n such an x_n

(b) Explain why no subsequence of (x_n) can be bounded above.

(c) Explain why no subsequence of (x_n) can converge.

This contradicts our hypothesis that every sequence has a convergent subsequence, therefore X is bounded above.

(d) Explain why X must also be bounded below.

3. (proof continued) Next, suppose that a is an accumulation point of X. As we showed in class, there is a sequence (x_n) from X that converges to a.

(e) Give a proof of the fact that if a is an accumulation point of X, then there is a sequence from X converging to x.

(f) Explain why our hypotheses about X in the statement of the **Theorem** allows us to conclude that $a \in X$.

(g) Explain why we may we now conclude that X closed.

(h) May we now conclude that X is compact? Explain.

4. (10 points) Use the definition of the derivative to prove that if $f(x) = 1/x$ then $f'(2) = -1/4$.

5. (10 points) Suppose that f is a function and a, b and c are constants. Define $g(x) = a \cdot f(x) + b$ for all x in the domain of f. Use the definition of continuity to prove that if f is continuous at c then g is also continuous at c. (hint: consider the case $a = 0$ separately).

6. (9 points) State the three important theorems from Calculus that we proved this term: The Mean Value Theorem, The Intermediate Value Theorem and the Extreme Value Theorem.

7. (6 points) Give the proof of any one of these.

8. (10 points) Use the Mean Value Theorem to explain why if $f'(x) \geq 0$ on an interval I, then f is increasing on that interval.

9. (4 points) State Taylor's Theorem.

10. (3 points) Find the Taylor Polynomial for e^x of degree 2 and give the error/remainder term for this Taylor Polynomial.

11. (3 points) Use it to approximate e and find a bound on the error in this approximation.