MATH 1200: Review for Final

1. Prove that for any positive integer number n, $n^3 + 2n$ is divisible by 3.

2. Show that if $a_n = 2a_{n-1} + (-1)^n$ and $a_0 = 2$ then show that $a_n = (5 \cdot 2^n + (-1)^n)/3$.

3. Define for $x, y \in \mathbb{C}$, $x \sim y$ if there exists an $a \in \mathbb{R}$ with $a \neq 0$ and $ax = y$. The symbol $\sim$ is known as a relation on $\mathbb{C}$.

   (a) Show that $x \sim x$ for all $x \in \mathbb{C}$. (that is, show that $\sim$ is reflexive).

   (b) Show that for any $x, y \in \mathbb{C}$, if $x \sim y$, then $y \sim x$ (that is, show that $\sim$ is symmetric).

   (c) Show that for any $x, y, z \in \mathbb{C}$, if $x \sim y$ and $y \sim z$, then $x \sim z$ (that is, show that $\sim$ is transitive).

   (d) Let $C_x = \{y \in \mathbb{C} : x \sim y\}$. On a graph of the complex plane, draw all of the points in C_0. On a separate graph of the complex plane draw all of the points in C_1 and all of the points in C_{1+i}.

4. Determine which of the following relations on the given set are equivalence relations and which are not. Prove your claims.

   (a) On the set of real numbers $\mathbb{R}$, define the relation $R = \{(x, y) : x, y \in \mathbb{R}, x - y \in \mathbb{Z}\}$

   (b) On the set of integers $\mathbb{Z}$, define the relation $R = \{(x, y) : x, y \in \mathbb{Z}, x + y \text{ is even}\}$

   (c) On the set of complex numbers $\mathbb{C}$, define the relation

$$R = \{(x, y) : x, y \in \mathbb{C}, Re(x) \leq Re(y) \vee Im(x) \leq Im(y)\}$$

5. Find integers r and s such that $309r + 1234s = 1$.

6. Find all solutions to the equation $309x \equiv 5 \pmod{1234}$.

7. For the following statements, either prove that they are true or provide a counterexample:

   (a) Let $a, b, m, n \in \mathbb{Z}$ such that $m, n > 1$ and $n \mid m$. If $a \equiv b \pmod{m}$, then $a \equiv b \pmod{n}$

   (b) Let $a, b, c, m \in \mathbb{Z}$ such that $m > 1$. If $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m}$

   (c) Let $a, b, c, d, m \in \mathbb{Z}$ such that $c, d \geq 1$ and $m > 1$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a^c \equiv b^d \pmod{m}$

8. Show that for $n \geq 1$,

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1} .$$

9. For all integers $n \geq 2$, $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$.

10. Prove or disprove the following statements:

    (a) Let a and n be integers, if $a|n$, then $a|n^2$.

    (b) Let a and n be integers, if $a|n^2$, then $a|n$.

    (c) Let a, n, m be integers, if $a|n$ or $a|m$, then $a|(n \cdot m)$,.

    (d) Let a, n, m be integers, if $a|n$ and $a|m$, then $a|(n \cdot m)$,.

    (e) Let a, n, m be integers, if $a|(n \cdot m)$, then $a|n$ or $a|m$.

    (f) Let a, n, m be integers, if $a|(n \cdot m)$, then $a|n$ and $a|m$.

    (g) Let a, n, m be integers and $gcd(n, m) = 1$, if $a|(n \cdot m)$, then $a|n$ or $a|m$.

    (h) Let a, n, m be integers and $gcd(a, n) = 1$, if $a|(n \cdot m)$, then $a|n$ or $a|m$.

    (i) Let a, n, m be integers and assume that there are integers k, ℓ such that $ak + n\ell = m$, then $gcd(a, n)|m$.

11. Find all $z \in \mathbb{C}$ such that $z^2 = i$.

12. Let x, y be real numbers. For the following statements, either prove that they are true or provide a counterexample:

    (a) If $x + y$ is irrational, then at least one of x or y is irrational.

    (b) If $x + y$ is rational, then both x and y are rational.

    (c) Between any two rational numbers there is a rational number.

    (d) For all real numbers x, there is a y such that $x \cdot y$ is rational.

    (e) For all real numbers x, there is a y such that $x + y$ is an integer.

13. For which values of n is $(1 - i)^n$ real? What values of n make it imaginary?